Lecture 6: Block Adaptive Filters and Frequency Domain Adaptive Filters

Overview

- Block Adaptive Filters
  - Iterating LMS under the assumption of small variations in $\underline{w}(n)$
  - Approximating the gradient by time averages
  - The structure of the Block adaptive filter
  - Convergence properties
- Frequency Domain Adaptive Filters
  - Frequency domain computation of linear convolution
  - Frequency domain computation of linear correlation
  - Fast LMS algorithm
  - Improvement of convergence rate
  - Unconstrained frequency domain adaptive filtering
  - Self-orthogonalizing adaptive filters

Reference: Chapter 7 from Haykin's book *Adaptive Filter Theory* 2002

LMS algorithm

Given
- the (correlated) input signal samples $\{u(1), u(2), u(3), \ldots\}$, generated randomely;
- the desired signal samples $\{d(1), d(2), d(3), \ldots\}$ correlated with $\{u(1), u(2), u(3), \ldots\}$

1 Initialize the algorithm with an arbitrary parameter vector $\underline{w}(0)$, for example $\underline{w}(0) = 0$.

2 Iterate for $n = 0, 1, 2, 3, \ldots, n_{max}$

2.0	Read /generate a new data pair,	$(\underline{u}(n), d(n))$
2.1	(Filter output)	$y(n) = \underline{w}(n)^T \underline{u}(n) = \sum_{i=0}^{M-1} w_i(n) u(n-i)$
2.2	(Output error)	$e(n) = d(n) - y(n)$
2.3	(Parameter adaptation)	$\underline{w}(n+1) = \underline{w}(n) + \mu \underline{u}(n) e(n)$

□

Complexity of the algorithm: $2M + 1$ multiplications and $2M$ additions per iteration

The error signal $e(n)$ is computed using the parameters $\underline{w}(n)$, and we emphasize this by denoting $e_{\underline{w}(n)}(n)$.

Iterating LMS under the assumption of small variations in $\underline{w}(n)$

The new parameters in LMS are evaluated at each time step

$$\begin{aligned}\underline{w}(n+L) &= \underline{w}(n+L-1) + \mu\underline{u}(n+L-1)e_{\underline{w}(n+L-1)}(n+L-1) \\ &= \underline{w}(n+L-2) + \mu\underline{u}(n+L-2)e_{\underline{w}(n+L-2)}(n+L-2) + \mu\underline{u}(n+L-1)e_{\underline{w}(n+L-1)}(n+L-1) \\ &= \underline{w}(n) + \sum_{i=0}^{L-1} \mu\underline{u}(n+i)e_{\underline{w}(n+i)}(n+i)\end{aligned}$$

If the variations of parameters $\underline{w}(n+L-i)$ during the L steps of adaptation are small, $\underline{w}(n+L-i) \approx \underline{w}(n)$

$$\underline{w}(n+L) \approx \underline{w}(n) + \sum_{i=0}^{L-1} \mu\underline{u}(n+i)e_{\underline{w}(n)}(n+i)$$

Introduce a second time index k such that $n = kL$ with a fixed integer L

$$\underline{w}(kL+L) = \underline{w}((k+1)L) = \underline{w}(kL) + \mu \sum_{i=0}^{L-1} \underline{u}(n+i)e_{\underline{w}(n)}(n+i)$$

If the parameters are changed only at moments kL, we may change the notation $\underline{w}(k) \leftarrow \underline{w}(kL)$

$$\boxed{\underline{w}(k+1) = \underline{w}(k) + \mu \sum_{i=0}^{L-1} \underline{u}(kL+i)e_{\underline{w}(k)}(kL+i)}$$

The output of the filter is

$$\underline{y}(kL+i) = \underline{w}^T(k)\underline{u}(kL+i) \qquad i \in \{0, \ldots, L-1\}$$

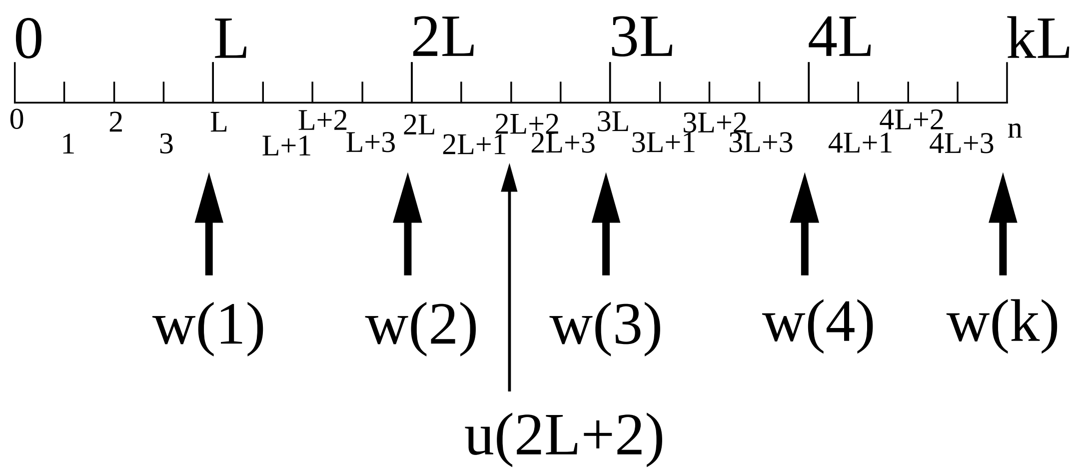
Block processing

Data used for modifying the partameters is grouped in blocks of length L.

The variables defined at time instants $n = kL + i$:

- the input signal $u(kL + i)$
- the output of the filter $y(kL + i) = \underline{w}^T(k)\underline{u}(kL + i)$
- the error signal $e(kL + i)$

The parameter vector, $\underline{w}(k)$, is defined only at time instants kL .



Block LMS algorithm

Given
- the (correlated) input signal samples $\{u(1), u(2), u(3), \ldots\}$, randomly generated;
- the desired signal samples $\{d(1), d(2), d(3), \ldots\}$ correlated with $\{u(1), u(2), u(3), \ldots\}$

1 Initialize the algorithm with an arbitrary parameter vector $\underline{w}(0)$, for example $\underline{w}(0) = 0$.

2 Iterate for $k = 0, 1, 2, 3, \ldots, k_{max}$ (k is the block index)

2.0 Initialize $\underline{\phi} = 0$

2.1 Iterate for $i = 0, 1, 2, 3, \ldots, (L-1)$

2.1.0 Read /generate a new data pair, $(\underline{u}(kL+i), d(kL+i))$

2.1.1 (Filter output) $y(kL+i) = \underline{w}(k)^T \underline{u}(kL+i) = \sum_{j=0}^{M-1} w_j(k) u(kL+i-j)$

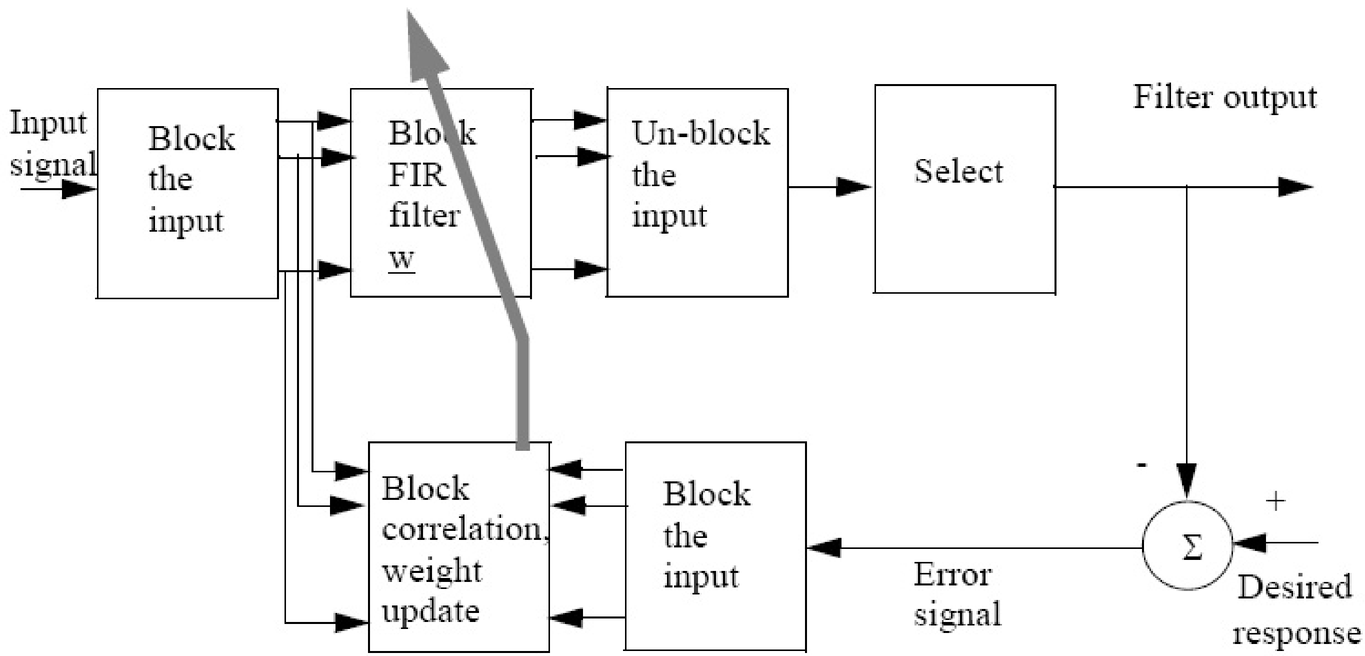
2.1.2 (Output error) $e(kL+i) = d(kL+i) - y(kL+i)$

2.1.3 (Accumulate) $\underline{\phi} \leftarrow \underline{\phi} + \mu e(kL+i) \underline{u}(kL+i)$

2.2 (Parameter adaptation) $\underline{w}(k+1) = \underline{w}(k) + \underline{\phi}$

□

Complexity of the algorithm: $2M+1$ multiplications and $2M + \frac{M}{L}$ additions per iteration


Input
signal
Block
the
input
Block
FIR
filter
w
Un-block
the
input
Select
Filter output
Block
correlation,
weight
update
Block
the
input
Error
signal
-
+
Σ
Desired
response

Another way to introduce Block LMS algorithm: approximating the gradient by time averages

The criterion

$$J = Ee^2(n) = E(d(n) - \underline{w}(n)^T \underline{u}(n))^2$$

has the gradient with respect to the parameter vector $\underline{w}(n)$

$$\nabla_{\underline{w}(n)} J = -2Ee(n)\underline{u}(n)$$

The adaptation of parameters in the Block LMS algorithm is

$$\underline{w}(k+1) = \underline{w}(k) + \mu \sum_{i=0}^{L-1} \underline{u}(kL+i) e_{\underline{w}(k)}(kL+i)$$

and denoting $\mu_B = \mu L$, the adaptation can be rewritten

$$\boxed{\underline{w}(k+1) = \underline{w}(k) + \mu_B \left[\frac{1}{L} \sum_{i=0}^{L-1} \underline{u}(kL+i) e_{\underline{w}(k)}(kL+i) \right] = \underline{w}(k) - \mu_B \frac{1}{2} \hat{\nabla}_{w(k)} J}$$

where we denoted by

$$\hat{\nabla}_{w(k)} J = -\frac{1}{L} \sum_{i=0}^{L-1} \underline{u}(kL+i) e_{\underline{w}(k)}(kL+i)$$

which shows that expectation in the expression of the gradient is replaced by time average.

Convergence properties of the Block LMS algorithm:

- **Convergence of average parameter vector** $E\underline{w}(n)$

  We will subtract the vector $\underline{w}_o$ from the adaptation equation

$$\underline{w}(k+1) = \underline{w}(k) + \mu\frac{1}{L}\sum_{i=0}^{L-1}\underline{u}(kL+i)e_{\underline{w}(k)}(kL+i) = \underline{w}(k) + \mu\frac{1}{L}\sum_{i=0}^{L-1}\underline{u}(kL+i)(d(kL+i) - \underline{u}(kL+i)^T\underline{w}(k))$$

and we will denote $\underline{\varepsilon}(k) = \underline{w}(k) - \underline{w}_o$

$$\underline{w}(k+1) - \underline{w}_o = \underline{w}(k) - \underline{w}_o + \mu\frac{1}{L}\sum_{i=0}^{L-1}\underline{u}(kL+i)(d(kL+i) - \underline{u}(kL+i)^T\underline{w}(k))$$

$$\underline{\varepsilon}(k+1) = \underline{\varepsilon}(k) + \mu\frac{1}{L}\sum_{i=0}^{L-1}\underline{u}(kL+i)(d(kL+i) - \underline{u}(kL+i)^T\underline{w}_o) +$$

$$+\mu\frac{1}{L}\sum_{i=0}^{L-1}(\underline{u}(kL+i)\underline{u}(kL+i)^T\underline{w}_o - \underline{u}(kL+i)\underline{u}(kL+i)^T\underline{w}(k))$$

$$= \underline{\varepsilon}(k) + \mu\frac{1}{L}\sum_{i=0}^{L-1}\underline{u}(kL+i)e_o(kL+i) - \mu\frac{1}{L}\sum_{i=0}^{L-1}\underline{u}(kL+i)\underline{u}(kL+i)^T\underline{\varepsilon}(k)$$

$$= \quad (I - \mu\frac{1}{L}\sum_{i=0}^{L-1}\underline{u}(kL+i)\underline{u}(kL+i)^T)\underline{\varepsilon}(k) + \mu\frac{1}{L}\sum_{i=0}^{L-1}\underline{u}(nkL+i)e_o(kL+i)$$

Taking the expectation of $\underline{\varepsilon}(k+1)$ using the last equality we obtain

$$E\underline{\varepsilon}(k+1) \quad = \quad E(I - \mu\frac{1}{L}\sum_{i=0}^{L-1}\underline{u}(kL+i)\underline{u}(kL+i)^T)\underline{\varepsilon}(k) + E\mu\frac{1}{L}\sum_{i=0}^{L-1}\underline{u}(nkL+i)e_o(kL+i)$$

and now using the statistical independence of $\underline{u}(n)$ and $\underline{w}(n)$, which implies the statistical independence of $\underline{u}(n)$ and $\underline{\varepsilon}(n)$,

$$E\underline{\varepsilon}(k+1) = (I - \mu E[\frac{1}{L}\sum_{i=0}^{L-1}\underline{u}(kL+i)\underline{u}(kL+i)^T])E[\underline{\varepsilon}(k)] + \mu E[\frac{1}{L}\sum_{i=0}^{L-1}\underline{u}(nkL+i)e_o(kL+i)]$$

Using the principle of orthogonality which states that $E[\underline{u}(kL+i)e_o(kL+i)] = 0$, the last equation becomes

$$E[\underline{\varepsilon}(k+1)] = (I - \mu E[\underline{u}(kL+i)\underline{u}(kL+i)^T])E[\underline{\varepsilon}(k)] = (I - \mu R)E[\underline{\varepsilon}(k)]$$

Reminding the equation

$$\underline{c}(n+1) = (I - \mu R)\underline{c}(n) \tag{1}$$

which was used in the analysis of SD algorithm stability, and identifying now $\underline{c}(n)$ with $E\underline{\varepsilon}(n)$, we have the following result:

The mean $E\underline{\varepsilon}(k)$ converges to zero, and consequently $E\underline{w}(k)$ converges to $\underline{w}_o$

iff

$0 < \mu < \dfrac{2}{\lambda_{max}}$ $(STABILITY CONDITION!)$ where λ_{max} is the largest eigenvalue of the matrix $R = E[\underline{u}(n)\underline{u}(n)^T]$.

Stated in words, block LMS algorithm is convergent in mean, iff the stability condition is met.

Study using small-step assumption

- The average time constant is

$$\tau_{mse,av} = \frac{L}{2\mu_B \lambda_{av}} \tag{2}$$

where λ_{av} is the average of the M eigenvalues of the correlation matrix

$$R \;=\; E[\underline{u}(n)\underline{u}^T(n)] \tag{3}$$

To compare, the average time constant for standard LMS is

$$\tau_{mse,av} = \frac{1}{2\mu \lambda_{av}} \tag{4}$$

therefore, the transients have the same convergence speed for block and standard LMS.

- Misadjustment The misadjustment

$$\mathcal{M} \triangleq \frac{J(\infty) - J_{min}}{J_{min}} = \frac{\mu_B}{2L} tr[R] \tag{5}$$

(where J_{min} is the MSE of the optimal Wiener filter) is the same as for the standard LMS algorithm.

- **Choice of block size**

  In most application the block size is selected to be equal to the filter length $L = M$. It is a tradeoff of the following drawbacks:

  – For $L > M$ the gradient is estimated using more data than the filter itself.

  – For $L < M$ the data in the current block is not enough to feed the whole tap vector, and consequently some weights are not used.

Frequency Domain Adaptive Filters

- FFT domain computation of the linear convolution with *Overlap-Save* method

  We want to compute simultaneously all the outputs of the block filter, corresponding to one block of data. Note that the filter parameters are kept constant during a block processing.

$$y(kM+m) = \sum_{i=0}^{M-1} w_i u(kM+m-i)$$

$$\begin{aligned}
y(kM) &= \sum_{i=0}^{M-1} w_i u(kM-i) = w_0 u(kM) + w_1 u(kM-1) + \ldots + w_{M-1} u(kM-M+1) \\
y(kM+1) &= \sum_{i=0}^{M-1} w_i u(kM-i+1) = w_0 u(kM+1) + w_1 u(kM) + \ldots + w_{M-1} u(kM-M+2) \\
y(kM+2) &= \sum_{i=0}^{M-1} w_i u(kM-i+2) = w_0 u(kM+2) + w_1 u(kM+1) + \ldots + w_{M-1} u(kM-M+3) \\
&\ldots \\
y(kM+(M-1)) &= \sum_{i=0}^{M-1} w_i u(kM-i+(M-1)) = w_0 u(kM+(M-1)) + w_1 u(kM+(M-2)) + \ldots + w_{M-1} u(kM)
\end{aligned}$$

  Let us consider two FFT transformed sequences:

  - the M-length weight vector is padded at the end with M zeros and then a $2M$-length FFT is computed

$$\underline{W} = FFT \begin{bmatrix} \underline{w} \\ 0 \end{bmatrix}$$

or componentwise:

$$W_i = \sum_{n=0}^{M-1} w(n) e^{-j\frac{2\pi i n}{2M}}$$

– the FFT transform of the vector $\underline{u} = [u(kM-M)\ \ u(kM-M+1)\ \ \ldots\ \ u(kM)\ \ u(kM+1)\ \ \ldots\ \ u(kM+M-1)]$ is then computed

$$U_i = \sum_{\ell=0}^{2M-1} u(kM - M + \ell) e^{-j\frac{2\pi i \ell}{2M}}$$

We try to rewrite in a different form the product of the terms $W_i U_i$ for $i = 0, \ldots, 2M-1$:

$$\begin{aligned}
W_i U_i &= \sum_{n=0}^{M-1} w(n) e^{-j\frac{2\pi i n}{2M}} \sum_{\ell=0}^{2M-1} u(kM - M + \ell) e^{-j\frac{2\pi i \ell}{2M}} = \sum_{n=0}^{M-1} \sum_{\ell=0}^{2M-1} w(n) u(kM - M + \ell) e^{-j\frac{2\pi i (n+\ell)}{2M}} \\
&= e^{-j\frac{2\pi i (M)}{2M}} \sum_{n=0}^{M-1} w(n) u(kM - n) + e^{-j\frac{2\pi i (M+1)}{2M}} \sum_{n=0}^{M-1} w(n) u(kM - n + 1) + \ldots + \\
&\quad + e^{-j\frac{2\pi i (M+M-1)}{2M}} \sum_{n=0}^{M-1} w(n) u(kM - n + M - 1) + \left(e^{-j\frac{2\pi i (0)}{2M}} C_0 + \ldots + e^{-j\frac{2\pi i (M-1)}{2M}} C_{M-1} \right) \\
&= e^{-j\frac{2\pi i (M)}{2M}} y(kM) + e^{-j\frac{2\pi i (M+1)}{2M}} y(kM+1) + \ldots + e^{-j\frac{2\pi i (2M-1)}{2M}} y(kM + M - 1) + \\
&\quad + \left(e^{-j\frac{2\pi i (0)}{2M}} C_0 + \ldots + e^{-j\frac{2\pi i (M-1)}{2M}} C_{M-1} \right) = \text{the } i\text{th element of } FFT \begin{bmatrix} \underline{C} \\ \underline{y}(kM) \end{bmatrix}
\end{aligned}$$

Denoting $\underline{y} = [y(kM)\ \ y(kM+1)\ \ \ldots\ \ y(kM+M-1)]^T$, we obtain finally the identity:

$$\begin{bmatrix} \underline{C} \\ \underline{y} \end{bmatrix} = IFFT \left(FFT \left(\begin{bmatrix} \underline{w} \\ 0 \end{bmatrix} \right) \times FFT \left(\begin{bmatrix} \underline{u} \end{bmatrix} \right) \right)$$

where by $\times$ we denoted the element-wise product of the vectors.

- FFT domain computation of the linear correlation

  We want to compute simultaneously all entries in the correlation vector needed in the adaptation equation

$$\underline{\phi} = \sum_{i=0}^{M-1} e(kM+i)\underline{u}(kM+i) = \sum_{i=0}^{M-1} \begin{bmatrix} u(kM+i) \\ u(kM+i-1) \\ . \\ . \\ u(kM+i-(M-1)) \end{bmatrix} e(kM+i)$$

$$\begin{aligned} \phi_\ell &= \sum_{i=0}^{M-1} e(kM+i)u(kM+i-\ell) \\ \phi_0 &= \sum_{i=0}^{M-1} e(kM+i)u(kM+i) = e(kM)u(kM) + \ldots + e(kM+M-1)u(kM+M-1) \\ \ldots & \\ \phi_{M-1} &= \sum_{i=0}^{M-1} e(kM+i)u(kM+i-(M-1)) \end{aligned}$$

  Let us consider the following FFT transformed sequence:

  – the M-length error vector $\underline{e} = [e(kM) \;\; e(kM+1) \;\; \ldots \;\; e(kM+(M-1))]^T$ is padded at the beginning with M zeros and then a $2M$-length FFT is computed

$$\underline{E} = FFT \begin{bmatrix} 0 \\ \underline{e} \end{bmatrix}$$

or componentwise:

$$E_i = \sum_{n=0}^{M-1} e(kM+n)e^{-j\frac{2\pi i(n+M)}{2M}} \qquad U_i = \sum_{\ell=0}^{2M-1} u(kM-M+\ell)e^{-j\frac{2\pi i\ell}{2M}}$$

We try to rewrite in a different form the product of the terms $E_i\overline{U}_i$ for $i = 0, \ldots, 2M-1$:

$$\begin{aligned}
E_i\overline{U}_i &= \sum_{n=0}^{M-1} e(kM+n)e^{-j\frac{2\pi i(n+M)}{2M}} \sum_{\ell=0}^{2M-1} u(kM-M+\ell)e^{j\frac{2\pi i\ell}{2M}} = \sum_{n=0}^{M-1}\sum_{\ell=0}^{2M-1} e(kM+n)u(kM-M+\ell)e^{-j\frac{2\pi i(n+M-\ell)}{2M}} \\
&= e^{-j\frac{2\pi i(M-1)}{2M}} \sum_{n=0}^{M-1} e(kM+n)u(kM+n-(M-1)) + e^{-j\frac{2\pi i(M-2)}{2M}} \sum_{n=0}^{M-1} e(kM+n)u(kM+n-(M-2)) + \ldots + \\
&\quad + e^{-j\frac{2\pi i(0)}{2M}} \sum_{n=0}^{M-1} e(kM+n)u(kM+n) + \left(e^{-j\frac{2\pi i(M)}{2M}} D_M + \ldots + e^{-j\frac{2\pi i(2M-1)}{2M}} D_{2M-1}\right) \\
&= e^{-j\frac{2\pi i(0)}{2M}}\phi_0 + e^{-j\frac{2\pi i(1)}{2M}}\phi_1 + \ldots + e^{-j\frac{2\pi i(M-1)}{2M}}\phi_{M-1} + \left(e^{-j\frac{2\pi i(M)}{2M}} D_M + \ldots + e^{-j\frac{2\pi i(2M-1)}{2M}} D_{2M-1}\right) \\
&= \text{the } i\text{th element of } FFT\begin{bmatrix} \underline{\phi} \\ \underline{D} \end{bmatrix}
\end{aligned}$$

We obtained finally the identities:

$$FFT\begin{bmatrix} \underline{\phi} \\ \underline{D} \end{bmatrix} = FFT\left(\begin{bmatrix} \underline{0} \\ \underline{e} \end{bmatrix}\right) \times \overline{FFT\left(\begin{bmatrix} \underline{u} \end{bmatrix}\right)} \quad \text{and} \quad \begin{bmatrix} \underline{\phi} \\ \underline{D} \end{bmatrix} = IFFT\left(FFT\left(\begin{bmatrix} \underline{0} \\ \underline{e} \end{bmatrix}\right) \times \overline{FFT\left(\begin{bmatrix} \underline{u} \end{bmatrix}\right)}\right)$$

where by $\times$ we denoted the element-wise product of the vectors.

The adaptation equation

$$\underline{w}(k+1) = \underline{w}(k) + \mu \sum_{i=0}^{M-1} \underline{u}(kM+i) e_{\underline{w}(k)}(kM+i) = \underline{w}(k) + \mu \underline{b}\phi$$

Due to linearity of FFT, we can write

$$FFT \begin{bmatrix} \underline{w}(k+1) \\ \underline{0} \end{bmatrix} = FFT \begin{bmatrix} \underline{w}(k) \\ \underline{0} \end{bmatrix} + \mu FFT \begin{bmatrix} \underline{\phi} \\ \underline{0} \end{bmatrix}$$

The fast LMS algorithm (Frequency Domain Adaptive Filter=FDAF

For each block of M data samples do the following:

1 Compute the output of the filter for the block $kM, \ldots, kM + M - 1$

$$\begin{bmatrix} \underline{C} \\ \underline{y} \end{bmatrix} = IFFT \left(FFT \left(\begin{bmatrix} \underline{w}(k) \\ 0 \end{bmatrix} \right) \times FFT \left(\begin{bmatrix} \underline{u} \end{bmatrix} \right) \right)$$

2 Compute the correlation vector

$$\begin{bmatrix} \underline{\phi} \\ \underline{D} \end{bmatrix} = IFFT \left(FFT \left(\begin{bmatrix} \underline{0} \\ \underline{e} \end{bmatrix} \right) \times \overline{FFT \left(\begin{bmatrix} \underline{u} \end{bmatrix} \right)} \right)$$

3 Update the parameters of the filter

$$FFT \begin{bmatrix} \underline{w}(k+1) \\ \underline{0} \end{bmatrix} = FFT \begin{bmatrix} \underline{w}(k) \\ \underline{0} \end{bmatrix} + \mu FFT \begin{bmatrix} \underline{\phi} \\ \underline{0} \end{bmatrix}$$

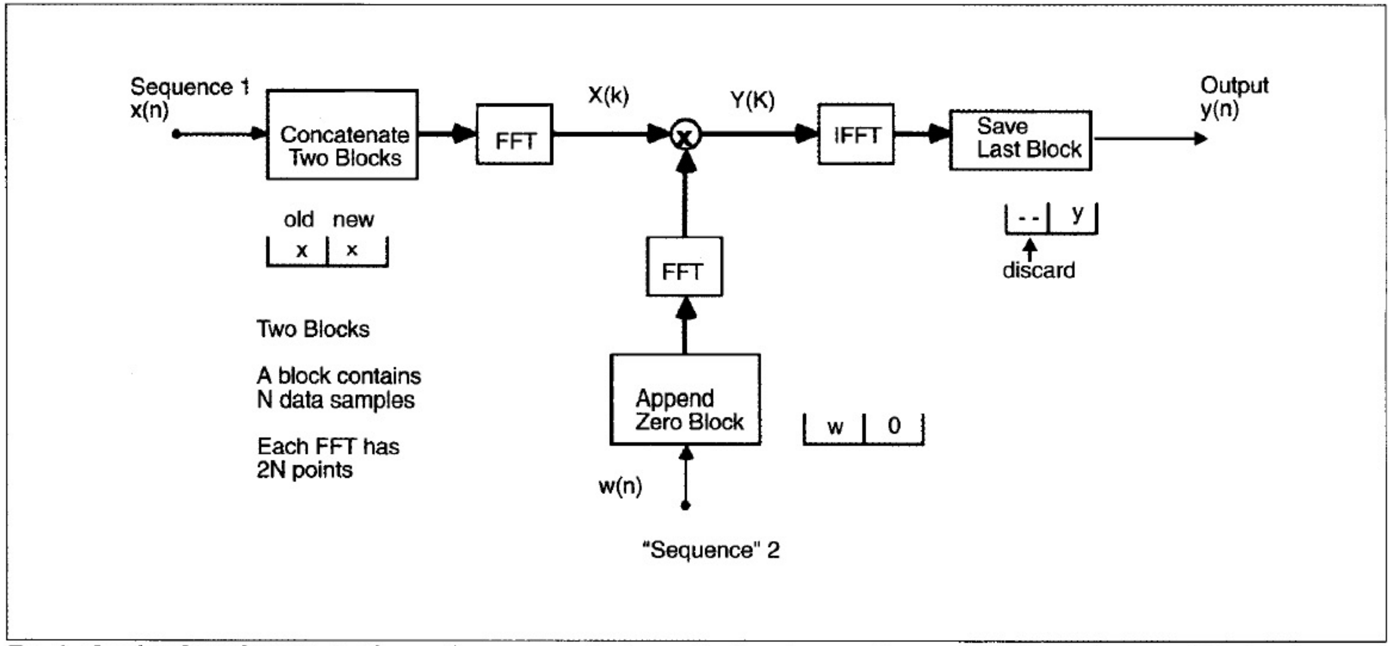


*Fig. 4. Overlap-Save Sectioning. The overlap-save sectioning method performs a linear convolution between a finite-length sequence and an infinite-length sequence by appropriately partitioning the data. The finite-length "sequence" **w**(n) (in our case, the adaptive weights) has N elements; after appending N zeros, a 2N-point FFT is computed. For the infinite-length input sequence **x** (n), the most recent N data samples are concatenated with the previous block of N samples; a 2N-point DFT of this extended data vector is then computed. The product of the transformed sequences (i.e., **Y**(k) = **X** (k) **W** (k)) is processed by a 2N-point inverse FFT (IFFT), yielding a block of output samples. The first N points of this output frame are discarded, while the last N points are the desired output samples of a linear convolution.*

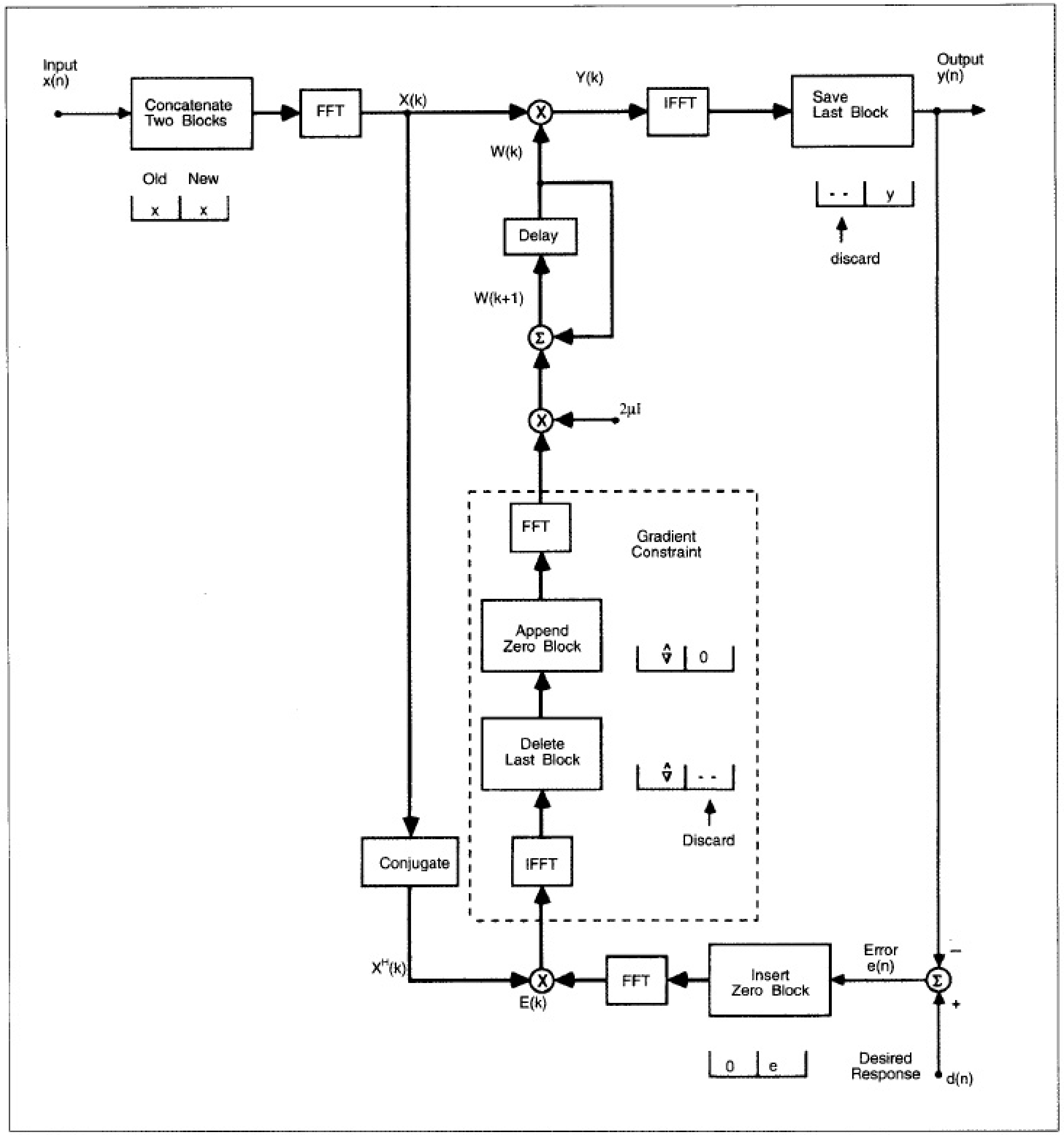


Fig. 5. Overlap-Save FDAF. This FDAF is based on the overlap-save sectioning procedure for implementing linear convolutions and linear correlations. The gradient constraint ensures that the IDFT of the 2N frequency-domain weights yields only N non-zero time-domain weights. Because the DFTs are computed only once for each block of data, there is an ***end-to-end*** *delay of N samples.*

Computational Complexity of the fast LMS algorithm

1 Classical LMS requires $2M$ multiplications per sample, so for a block of M samples there is a neeed of $2M^2$ multiplications.

2 In the fast LMS algorithm there are 5 FFT transforms , requiring approximately $2M\log(2M)$ real multiplications each, and also other $16M$ operations (when updating the parameters, computing the errors, element-wise multiplications of FFT transformed vectors) so the total is

$$10M\log(2M) + 16M = 10M\log(M) + 26M$$

3 The complexity ratio for the fast LMS to standard LMS is

$$\text{Complexity ratio} = \frac{2M^2}{10M\log(M) + 26M} = \frac{M}{5\log_2(M) + 13}$$

For $M = 16$ Complexity ratio=0.48 Classical LMS is superior

For $M = 32$ Complexity ratio=0.84 Classical LMS is superior

For $M = 64$ Complexity ratio=1.49 Frequency domain LMS is superior

For $M = 1024$ Complexity ratio=16 Frequency domain LMS is 16 times faster than classical LMS

For $M = 2048$ Complexity ratio=30 Frequency domain LMS is 30 times faster than classical LMS

Convergence rate improvement

- In fast LMS, since the weights are adapted in the frequency domain, they can be associated to one mode of the adaptive process. The individual convergence rate may be varied in a straightforward manner. This is different of the mixture of modes type of adaptation, which was found in LMS.
- The convergence time for the i'th mode is inversely proportional to $\mu\lambda_i$, where λ_i is the eigenvalue of the correlation matrix R of the input vector, and λ_i is a measure of the average input power in the i'th frequency bin.
- All the modes will converge at the same rate by assigning to each weight a different step-size

  $$\mu_i = \frac{\alpha}{P_i}$$

  where P_i is an estimate of the average power in the i'th bin, and α controls the overall time constant of the convergence process

  $$\tau = \frac{2M}{\alpha}\text{samples}$$

  If the environment is non-stationary, the estimation of P_i can be carried out by

  $$P_i(k) = \gamma P_i(k-1) + (1-\gamma)|U_i(k)|^2, \qquad i = 0, 1, \ldots, 2M-1$$

  where γ is a forgetting factor

Unconstrained frequency-domain adaptive filtering

- In the computation of the gradient, some constraints are imposed in order to achieve a linear correlation, (as opposed to a a circular correlation). These constraints are:
  * Discard the last M elements of the inverse FFT of $\underline{U}^H(k)\underline{E}(k)$
  * Replace the elements discarded by an appended block of zeros.
- If from the flow-graph of the LMS algorithm the gradient constraints are removed (a FFT block, a IFFT block, the delete block, and the append block), the algorithm is no longer equivalent to block LMS block

$$\underline{W}(k+1) = \underline{W}(k) + \mu \underline{U}^H(k)\underline{E}(k) \tag{6}$$

- The resulting algorithm has a lower complexity (only three FFTs are required).
- The drawbacks:
  * when the number of processed blocks increases, the weight vector no longer converges to the Wiener solution.
  * the steady state error of the unconstrained algorithm is increased compared to the fast LMS algorithm.

Self-orthogonalizing adaptive filters

The self-orthogonizing adaptive filter was introduced to guarantee a constant convergence rate, not dependent on the input statistics.

– The updating equation is

$$\underline{w}(n+1) = \underline{w}(n) + \alpha R^{-1}\underline{u}(n)e(n)$$

– the step size must satisfy $0 < \alpha < 1$ and it was recommended to be selected as

$$\alpha = \frac{1}{2M}$$

– Example: for white Gaussian *input*, with variance σ^2,

$$R = \sigma^2 I$$

and the adaptation becomes the one from the standard LMS algorithm:

$$\underline{w}(n+1) = \underline{w}(n) + \frac{1}{2M\sigma^2}\underline{u}(n)e(n)$$

– From the previous example, a two stage procedure can be inferred:

  * Step I: Transform the input vector $\underline{u}(n)$ into a corresponding vector of uncorrelated variables.
  * Step II: use the transformed vector into an LMS algorithm

– Consider first as uncorrelating transformation the Karhunen-Loeve transform:

$$\nu_i(n) = \underline{q}_i^T \underline{u}(n), \qquad , i = 0, \ldots, M-1$$

where $\underline{q}_i$ is the eigenvector associated with the i'th eigenvalue λ_i of the correlation matrix R of the input vector $\underline{u}(n)$.

- The individual outputs of the KLT are uncorrelated:

$$E\nu_i(n)\nu_j(n) = \begin{cases} \lambda_i, & j = i \\ 0, & j \neq i \end{cases}$$

- The adaptation equation (Step II) becomes

$$\underline{w}(n+1) = \underline{w}(n) + \alpha\Lambda^{-1}\underline{\nu}(n)e(n)$$

or written element-wise, for $i = 0, 1, \ldots, M-1$:

$$w_i(n+1) = w_i(n) + \frac{\alpha}{\lambda_i}\nu_i(n)e(n)$$

- Replacing the optimal KLT with the (sub)optimal DCT (discrete cosine transform) one obtains the DCT-LMS algorithm.
- The DCT is performed at each sample (the algorithm is no longer equivalent to a block LMS. Advantage: better convergence. Disadvantage: not so computationally efficient.